# INEQUALITIES

## UNIT 2 TECHNIQUES IN PROVING INEQUALITIES

In Unit 1 we learnt several classical inequalities. While the inequalities themselves can be easily memorised, it is often not easy to apply them to prove other inequalities in practical situations. A general method is to start from one side of the inequality and apply a sequence of known inequalities to reach the other side, or we may start from both sides and try to 'make ends meet'.

As a matter of fact, very few problems concerning inequalities can be proved by a direct application of one of the inequalities we have learned. Also, sometimes it is not too apparent which classical inequality to apply in dealing with a problem. Ingenious tricks are often needed to obtain a nice solution. Repeated trials and deep thoughts are also demanded.

In this unit we shall look at some techniques that are frequently employed in proving inequalities. However, it should always be remembered that there is no standard way of proof and there is no general rule in choosing which technique to use. Practice and experience, however, often help.

## 1. Suitable Applications of Classical Inequalities

We often have to substitute expressions into the classical inequalities we learned in Unit 1 to prove new inequalities. Different substitutions often yield different results. It is therefore important that we apply known inequalities suitably in order that the desired result can be obtained.

**Example 1.1.**

Prove that for positive real numbers *a*, *b*, *c*,

$$(a+b)(b+c)(c+a) \geq 8abc\,.$$

**Solution.**

As the left side involves sums and the right side involves products, it is natural that we try the AM-GM inequality. But the left side involves the product of three sums. How should the AM-GM inequality be applied?

Beginners may try to expand the left side so that the inequality becomes

$$a^2b+a^2c+b^2a+b^2c+c^2a+c^2b+2abc\ge 8abc\,.$$

The left side is now the sum of 7 terms, so applying the AM-GM inequality on this sum yields

$$a^2b+a^2c+b^2a+b^2c+c^2a+c^2b+2abc\ge 7\cdot\sqrt[7]{(a^2b)(a^2c)(b^2a)(b^2c)(c^2a)(c^2b)(2abc)}=7\cdot\sqrt[7]{2}abc\,.$$

The logic is completely correct. In fact this resulting inequality is true, but it is not what we want. So what's wrong?

If we cancel $2abc$ on both sides, the inequality becomes $a^2b+a^2c+b^2a+b^2c+c^2a+c^2b\ge 6abc$. Now applying AM-GM inequality gives us the desired result:

$$a^2b+a^2c+b^2a+b^2c+c^2a+c^2b\ge 6\sqrt[6]{(a^2b)(a^2c)(b^2a)(b^2c)(c^2a)(c^2b)}=6abc$$

Alternatively, writing $2abc$ as $abc+abc$ also gives us the desired result:

$$\begin{aligned}(a+b)(b+c)(c+a)&=a^2b+a^2c+b^2a+b^2c+c^2a+c^2b+abc+abc\\&\ge 8\cdot\sqrt[8]{(a^2b)(a^2c)(b^2a)(b^2c)(c^2a)(c^2b)(abc)(abc)}\\&=8abc\end{aligned}$$

In fact, the easiest way is not to expand the product at all and apply the AM-GM inequality right the way:

$$(a+b)(b+c)(c+a)\ge\left(2\sqrt{ab}\right)\left(2\sqrt{bc}\right)\left(2\sqrt{ca}\right)=8abc\,.$$

The previous example shows that with the same expressions we can apply inequalities in many different ways, and they end up in different results. Of course we may not be so lucky to have applied the inequality in the correct way the first time, in which case we need to make repeated trials.

In general, experience shows that when attempting to prove a given inequality, expanding the terms grouped in a product is usually not the best way. There must be some reasons that the terms be written as a product of factors. Expanding (or sometimes squaring both sides involving radicals) by brute force will usually end up in a mess.

**Example 1.2.**

Prove that for positive real numbers *a*, *b*, *c*,

$$a^2+b^2+c^2 \geq ab+bc+ca\,.$$

**Solution.**

Clearly the inequality has a strong 'AM-GM feel'. But direct application of the AM-GM inequality does not work. Instead, we observe that the terms on the right hand side are products of two of *a*, *b*, *c*. Hence, we proceed as follows.

$$\begin{aligned} a^2+b^2+c^2 &= \frac{1}{2}(a^2+b^2)+\frac{1}{2}(b^2+c^2)+\frac{1}{2}(c^2+a^2) \\ &\geq \sqrt{a^2b^2}+\sqrt{b^2c^2}+\sqrt{c^2a^2} \\ &= ab+bc+ca \end{aligned}$$

This solution is nice, but it may not be easy to think of at the first very place. Perhaps it will be easier and more natural to start from the right hand side. To apply the AM-GM inequality on the term *ab*, there are two methods. One is to view *ab* as the geometric mean of $a^2$ and $b^2$, the other is to view *ab* as the square of the geometric mean of *a* and *b*. These two views of the term *ab* yield respectively (by applying the AM-GM inequality)

$$ab=\sqrt{a^2b^2} \leq \frac{1}{2}\left(a^2+b^2\right) \text{ and } ab=\left(\sqrt{ab}\right)^2 \leq \left(\frac{a+b}{2}\right)^2.$$

Here the first view works. These two upper bounds for *ab* should be well appreciated and familiarly handled. They are the most elementary applications of the AM-GM inequality and are often useful in proving more complicated inequalities.

## 2. Substitutions

Sometimes, an apparently complicated inequality becomes much easier by means of a suitable substitution.

**Example 2.1.**

(IMO 1995) Let *a*, *b*, *c* be positive real numbers such that *abc* = 1. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geq \frac{3}{2}\,.$$

**Solution.**

Let $x = ab$, $y = bc$ and $z = ca$. Then $x, y, z$ are positive real numbers such that $xyz = (abc)^2 = 1$.

The left hand side of original inequality becomes

$$\begin{aligned}\frac{1}{a^2(ab+ac)}+\frac{1}{b^2(bc+ba)}+\frac{1}{c^2(ca+cb)} &= \frac{y}{xz(x+z)}+\frac{z}{xy(x+y)}+\frac{x}{yz(y+z)} \\ &= \frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\left(\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}\right)\cdot\left[(y+z)+(x+z)+(x+y)\right] \geq (x+y+z)^2.$$

Hence

$$\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y} \geq \frac{(x+y+z)^2}{2(x+y+z)} = \frac{x+y+z}{2} = \frac{3}{2}\left(\frac{x+y+z}{3}\right) \geq \frac{3}{2}(xyz)^{\frac{1}{3}} = \frac{3}{2},$$

proving the original inequality.

As one could expect, there is no general rule telling what substitutions to make. We probably have to make repeated trials until we succeed. But sometimes there may be clues from the problem on the type of substitutions that we should try.

**Example 2.2.**

(IMO 2000) Let $a, b, c$ be positive real numbers such that $abc = 1$. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1.$$

**Solution.**

Since $abc = 1$, we could try the substitution

$$a = \frac{x}{y},\ b = \frac{y}{z},\ c = \frac{z}{x}$$

where $x, y, z$ are positive real numbers.

(Of course you may challenge that we also have $abc = 1$ in Example 2.1. Why don't we make such a substitution in that example? Well, this may not work all the time. Perhaps it works for the previous example; you may try.)

The original inequality becomes

$$\left(\frac{x}{y}-1+\frac{z}{y}\right)\left(\frac{y}{z}-1+\frac{x}{z}\right)\left(\frac{z}{x}-1+\frac{y}{x}\right)\leq 1.$$

Multiplying both sides by *xyz*, we get

$$(x-y+z)(y-z+x)(z-x+y)\leq xyz \qquad (*)$$

If the left hand side of (*) is negative, then we are done.

Otherwise, it suffices to show

$$(x-y+z)^2(y-z+x)^2(z-x+y)^2\leq x^2y^2z^2 \qquad (**)$$

Indeed, we have

$$\begin{aligned}(x-y+z)(y-z+x)&=x^2-(y-z^2)\leq x^2\\(y-z+x)(z-x+y)&=y^2-(z-x)^2\leq y^2\\(x-y+z)(z-x+y)&=z^2-(x-y)^2\leq z^2\end{aligned}$$

Multiplying these three inequalities together, we get (**), as desired.

The inequalities we have seen so far involve variables that are either real numbers or positive real numbers, sometimes with additional constraints like $abc = 1$. Sometimes an inequality may involve variables that are the lengths of the sides of a triangle. In this case we can use a special substitution technique.

**Example 2.3.**

(IMO 1983) Let *a*, *b* and *c* be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

Determine when equality holds.

**Solution.**

Note that *a*, *b*, *c* are the side lengths of a triangle, rather than just positive real numbers. We must use this condition somewhere in the solution (unless it is a deliberate trap, which is rare in large-scale competitions like the IMO).

Since *a*, *b*, *c* are the sides of a triangle, we can find positive real numbers *a*, *b*, *c* such that $a=y+z$, $b=z+x$ and $c=x+y$. (Considering the inscribed circle of the triangle, *x*, *y*, *z* are just the lengths of the tangents from the vertices to the circle.) Indeed, we have

$$x=\frac{b+c-a}{2},\ y=\frac{c+a-b}{2},\ z=\frac{a+b-c}{2}.$$

These numbers are easily seen to be positive in view of the triangle inequality.

With this substitution, the original inequality becomes

$$(y+z)^2(z+x)(y-x)+(z+x)^2(x+y)(z-y)+(x+y)^2(y+z)(x-z)\ge 0.$$

After simplification, we get

$$xy^3+yz^3+zx^3\ge xyz(x+y+z).$$

Dividing both sides by *xyz*, it becomes

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\ge x+y+z,$$

which easily follows from the fact that

$$x^2\ge y(2x-y),\ y^2\ge z(2y-z)\ \text{and}\ z^2\ge x(2z-x).$$

**Alternative Solution.**

(Due to Bernard Leeb, a West German contestant in IMO 1983, who won a special prize with this elegant solution.)

Without loss of generality, assume that *a* is the longest side of the triangle. Then

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)=a(b-c)^2(b+c-a)+b(a-b)(a-c)(a+b-c)\ge 0.$$

**Remark.** We may assume *a* to be the longest side of the triangle because a cyclic permutation of $(a,b,c)$ (i.e. $a\to b$, $b\to c$, $c\to a$, or $a\to c$, $c\to b$, $b\to a$, ) leaves the given inequality unchanged. However, we *cannot* assume $a\ge b\ge c$ because *a*, *b*, *c* are not symmetric. (e.g. switching *a* and *b* changes the inequality.)

## 3. Altering the number of variables

In general, an inequality involving fewer variables is easier to prove. For instance, an inequality in one variable may be proved by elementary algebraic methods like factorisation. Hence, if by some transformation the number of variables in an inequality can be reduced, it would be easier to prove the inequality.

**Example 3.1.**

(IMO 2001 Hong Kong Team Selection Test) Let *a*, *b*, *c* be positive real numbers. Prove that

$$(a+b)^2+(a+b+4c)^2 \ge \frac{100abc}{a+b+c}.$$

**Solution.**

Note that the inequality involves three variables, *a*, *b* and *c*. Among these, *a* and *b* are playing symmetric roles. Moreover, the left side is of degree 2, the right side is the quotient of a degree 3 polynomial by a degree 1 polynomial, which can also be viewed as degree 2.

To reduce the number of variables, we divide both sides of the inequality by $c^2$ and obtain

$$\left(\frac{a}{c}+\frac{b}{c}\right)^2+\left(\frac{a}{c}+\frac{b}{c}+4\right)^2 \ge \frac{100\cdot\frac{a}{c}\cdot\frac{b}{c}}{\frac{a}{c}+\frac{b}{c}+1}.$$

Clearly, if we set $x=\dfrac{a}{c}$ and $y=\dfrac{b}{c}$, we obtain the following inequality in two variables:

$$\left(x+y\right)^2+\left(x+y+4\right)^2 \ge \frac{100xy}{x+y+1}$$

Now *x* and *y* occur as $x+y$ except in the numerator of the right hand side. Hence we apply the fact

$$xy \le \left(\frac{x+y}{2}\right)^2.$$

Consequently, it will suffice to prove

$$(x+y+1)\left[(x+y)^2+(x+y+4)^2\right] \ge 25(x+y)^2.$$

It is clear that the substitution $z=x+y$ reduces the above inequality to the one-variable inequality

$$(z+1)\left[z^2+(z+4)^2\right] \ge 25z^2.$$

The last inequality is equivalent to the trivial inequality $(z-4)^2(2z+1)\ge 0$. Hence the original inequality is proved.

Sometimes an inequality may involve *n* variables, i.e. the number of variables is not fixed. To prove such inequalities, we may begin with the simple cases *n* = 1 and *n* = 2. These often give us inspirations and insights on how the general case may be proved.

**Example 3.2.**

Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$a_1^{a_1} a_2^{a_2} \cdots a_n^{a_n} \ge \left(a_1 a_2 \cdots a_n\right)^{\frac{a_1+a_2+\cdots+a_n}{n}}.$$

**Solution.**

When $n = 1$, the inequality becomes $a_1^{a_1} \ge a_1^{a_1}$, which is trivial.

When $n = 2$, the inequality becomes $a_1^{a_1} a_2^{a_2} \ge \left(a_1 a_2\right)^{\frac{a_1+a_2}{2}}$, which simplifies to $\left(\frac{a_1}{a_2}\right)^{\frac{a_1-a_2}{2}} \ge 1$.

Assuming (without loss of generality) that $a_1 \ge a_2$, then the base is at least 1 and the power is at least 0. The result follows.

When $n = 3$, the inequality becomes $a_1^{a_1} a_2^{a_2} a_3^{a_3} \ge \left(a_1 a_2 a_3\right)^{\frac{a_1+a_2+a_3}{3}}$. We try to simplify the inequality to a form analogous to that in the case $n = 2$. Indeed, the inequality simplifies to

$$\left(\frac{a_1}{a_2}\right)^{\frac{a_1-a_2}{3}} \left(\frac{a_1}{a_3}\right)^{\frac{a_1-a_3}{3}} \left(\frac{a_2}{a_3}\right)^{\frac{a_2-a_3}{3}} \ge 1.$$

Again, the result follows by assuming $a_1 \ge a_2 \ge a_3$.

Now, the proof of the general case is clear. Without loss of generality we assume $a_1 \ge a_2 \ge \cdots \ge a_n$. The inequality may be simplified as

$$\prod_{1 \le i < j \le n} \left(\frac{a_i}{a_j}\right)^{\frac{a_i-a_j}{n}} \ge 1$$

which is trivial under the ordering assumption.

While reducing the number of variables sometimes makes an inequality simpler, it may be surprising that some inequalities can be proved in an easy way by introducing additional variables. This is illustrated by the following example.

**Example 3.3.**

Prove that for positive real numbers $x$, $y$, $z$,

$$(x-y+z)(y-z+x)(z-x+y) \le xyz.$$

**Solution.**

Note that we have come across this somewhere in Example 2.2. We have seen a nice proof of this in Example 2.2, but here we present another proof under the additional assumption $x, y, z \ge 0$.

Without loss of generality assume $x \ge y \ge z$. Set $x = z + \delta_1$, $y = z + \delta_2$. Then $\delta_1 \ge \delta_2 \ge 0$. Hence

$$\begin{aligned} xyz - (x - y + z)(y - z + x)(z - x + y) &= (z + \delta_1)(z + \delta_2)z - (z + \delta_2 - \delta_1)(z + \delta_1 - \delta_2)(z + \delta_1 + \delta_2) \\ &= z\delta_1\delta_2 + (\delta_1 - \delta_2)^2 (z + \delta_1 + \delta_2) \\ &\ge 0 \end{aligned}$$

and the result follows.

## 4. Observing equality cases

An 'inequality' is an algebraic statement stating that two expressions are related by a certain relation, which can be represented using an inequality symbol. While 'inequality' means 'not equal', an inequality relation may carry an 'equality' component. For example, the symbol '≥' means 'greater than or equal to'.

As we have seen in Unit 1, different inequalities have different cases for equality. In the AM-GM inequality, equality occurs if and only if all variables are equal. In the rearrangement inequality, equality occurs when either the $a_i$ 's or the $b_i$ 's are equal. In the Cauchy-Schwarz inequality, equality occurs when the $a_i$ 's are proportional to the $b_i$ 's.

Since each inequality has its own instances of equality, observing cases of equality helps us determine which inequality to apply when attempting to prove a given inequality. Sometimes it also helps us determine in what way an inequality is to be applied.

**Example 4.1.**

(IMO 2001 Hong Kong Team Selection Test) Let *a*, *b*, *c* be positive real numbers. Prove that

$$(a+b)^2 + (a+b+4c)^2 \ge \frac{100abc}{a+b+c}.$$

**Solution.**

Note that this is the same inequality as in Example 3.1.

Here we try to proceed in another way. When does equality hold for this inequality?

Well, after some trial, we find that equality occurs when $a = b = 2c$. (The inspiration should come from the term $a + b + 4c$.) By the AM-GM inequality,

$$a+b+c=\left(\frac{a}{2}+\frac{a}{2}+\frac{b}{2}+\frac{b}{2}+c\right)\geq 5\cdot\sqrt[5]{\frac{a^2b^2c}{16}}.$$

Also,

$$\begin{aligned}(a+b)^2+(a+b+4c)^2 &= 2(a^2+b^2+8c^2+2ab+4ac+4bc)\\ &= 2(a^2+b^2+4c^2+4c^2+ab+ab+2ac+2ac+2bc+2bc)\\ &\geq 2\cdot 10\cdot\sqrt[10]{(a^2)(b^2)(4c^2)(4c^2)(ab)(ab)(2ac)(2ac)(2bc)(2bc)}\end{aligned}$$

Multiplying these together, we get the desired result.

Note that we split the terms in rather mysterious ways before the two applications of the AM-GM inequality. This is to ensure that in each application, the instances of equality coincide with the instances of equality of the given inequality.

**Example 4.2.**

Let $K(a,b,c)$ denote the area of a triangle with side lengths *a*, *b* and *c*. Show that

$$\sqrt{K(x,y,z)}+\sqrt{K(x',y',z')}\leq\sqrt{K(x+x',y+y',z+z')}.$$

**Solution.**

When faced with such a problem the most natural way to start is to experiment with certain values. We will probably start with some special types of triangles: we have $K(2,2,2)=\sqrt{3}$, $K(3,4,5)=6$, $K(6,8,10)=12$, $K(5,5,6)=12$, etc.

We then try to plug in these values into the inequality to check. We should find that the left hand side does not exceed the right hand side (if not either the problem is wrong or your substitution is wrong). But when does equality occur? Does it give you insight on which inequality to use?

We leave the proof of this inequality as an exercise.

## 6. Exercises

1. Let *x*, *y*, *z* be positive real numbers such that $z=x+y$. Prove that

$$\left(x^2+y^2+z^2\right)^3\geq 54x^2y^2z^2.$$

2. Let *x*, *y*, *z* be positive real numbers such that $x+y+z=1$. Prove that

$$xy + yz + zx - 3xyz \le \frac{1}{4}.$$

3. Complete the proof of the inequality in Example 4.2.

4. Let $a_1, a_2, \ldots, a_{2003}$ be positive real numbers such that

$$a_1 + a_2 + \cdots + a_{2003} < 1.$$

Determine the maximum possible value of

$$\frac{a_1 a_2 \cdots a_{2003}(1 - a_1 - a_2 - \cdots - a_{2003})}{(a_1 + a_2 + \cdots + a_{2003})(1 - a_1)(1 - a_2)\cdots(1 - a_{2003})}.$$

5. (APMO 1991) Let $a_1$, $a_2$, …, $a_n$, $b_1$, $b_2$, …, $b_n$ be positive real numbers such that

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k .$$

Show that

$$\sum_{k=1}^{n} \left( \frac{{a_k}^2}{a_k + b_k} \right) \ge \sum_{k=1}^{n} \left( \frac{a_k}{2} \right).$$

6. (APMO 1996) Let *a*, *b*, *c* be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \le \sqrt{a} + \sqrt{b} + \sqrt{c}$$

and determine when equality holds.

7. (IMO 1997 Hong Kong Team Selection Test) Prove that for positive real numbers *x*, *y*, *z*,

$$\frac{xyz\left(x + y + z + \sqrt{x^2 + y^2 + z^2}\right)}{\left(x^2 + y^2 + z^2\right)\left(xy + yz + zx\right)} \le \frac{3 + \sqrt{3}}{9}.$$

8. (USAMO 1997) Prove that, for positive real numbers *a*, *b* and *c*,

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \le \frac{1}{abc}.$$

9. (IMO 1998 Hong Kong Team Selection Test) Show that for positive real numbers *a*, *b*, *c* not less than 1,

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}\le\sqrt{c(ab+1)}\ .$$

(Hint: Let $a-1=x^2$, $b-1=y^2$, $c-1=z^2$.)

10. (IMO 1999) Let *n* be a fixed integer, with $n\ge 2$.

(a) Determine the least constant *C* such that the inequality

$$\sum_{1\le i<j\le n} x_i x_j (x_i^2+x_j^2)\le C\left(\sum_{1\le i\le n} x_i\right)^4$$

holds for all real numbers $x_1, \ldots, x_n \ge 0$.

(b) For this constant *C*, determine when equality holds.